2021

MATHEMATICS — HONOURS

Paper : CC-6

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative with proper justification (1 mark for correct answer and 1 mark for justification) : (1+1)×10

(a) The integral domain of matrices $\left\{\begin{pmatrix} a & b \\ 3b & a \end{pmatrix} : a, b \in \mathbb{Q}\right\}$, where $\mathbb{Q}$ is the set of all rationals, under matrix addition and multiplication is

(i) not a field (ii) a field

(iii) a skew field but not a field (iv) none of the above.

(b) Let A and B be ideals of a ring R. Choose from the following properties which holds for R

(i) $A + A = A$ (ii) $AB = A \cap B$, always for all ring R

(iii) $A + B = A \cup B$, always $\forall$ ring R (iv) none of the above.

(c) Which of the following statement is true?

(i) $(z, +, \cdot)$ has no unit element

(ii) In the ring $(z_6, +, \cdot)$, every non-zero element is unit

(iii) The set of all unit in a ring R with unity does not form a group w.r.t. multiplication

(iv) If a be a unit in a ring R with unity, then a is not a divisor of zero.

(d) The ideal $(2z, +, \cdot)$ in the ring $(z, +, \cdot)$, with z denoting set of all integers, is

(i) a prime ideal (ii) not a prime ideal

(iii) a maximal ideal but not a prime ideal (iv) none of the above.

(e) In a field of characteristic three, $(a + b)^6$ equals to

(i) $(a^3 + b^3)(a^2 + 2ab + b^2)$ (ii) $a^6 + b^6$

(iii) $(a^3 + b^3)(a^2 + b^2)$ (iv) none of these.

(f) The number of independent elements in Z_{mn} where $m > 1$, $n > 1$ are relatively prime is

(i) at least 2 (ii) at least 4

(iii) 0 (iv) none of the above.

(g) For the subspaces $U = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \right\}$ and $V = \left\{ \begin{pmatrix} a & b \\ c & -c \end{pmatrix} \right\}$ the dim U and dim V are

(i) (2, 1) (ii) (2, 3)

(iii) (2, 2) (iv) (1, 3)

(h) Which one is not a vector space?

(i) $\mathbb{C}$ over $\mathbb{C}$ (ii) $\mathbb{C}$ over $\mathbb{R}$

(iii) $\mathbb{R}$ over $\mathbb{C}$ (iv) $\mathbb{R}$ over $\mathbb{Q}$

(i) Which of the following is a subspace of $\mathbb{R}^3$?

(i) $2a + 3b + 5c = 0$ (ii) $2a + 3b + 5c = 1$

(iii) $2a + 3b + 5c = -1$ (iv) None of the above; where $(a, b, c) \in \mathbb{R}^3$

(j) For a real symmetric matrix, eigenvalues are

(i) all real (ii) all complex

(iii) cannot be ascertained (iv) both real and complex.

Unit - I

Answer ***any five*** questions.

2. (a) Let p be any prime integer, then show that there are only two non-isomorphic rings of p elements. 5

(b) Find the maximal ideals and prime ideals of the ring Z_8. 3+2

(c) If n is a positive integer and a is only prime integer to n, then $a^{\varphi(n)} \equiv 1 (\text{mod } n)$, where $\varphi(n)$ is number of +ve integers less than and prime to n. Prove using theory of rings. 5

(d) Prove that the subring $S = \left\{ a + b\sqrt{5};\, a, b \in Q \right\}$ is a subfield of $\mathbb{R}$ (Set of real numbers). 3+2

(e) If S and T are ideals of a ring R such that $S \cap T = \{0\}$, prove that $xy = 0\ \forall\ x \in s$ and $y \in T$. 5

(f) In a commutative ring R with unity, prove that an ideal V is a prime ideal iff the quotient ring R/V is an integral domain. 3+2

(g) State fundamental theorem of ring homomorphism. Show that a ring with unity is a field, by definition. 2+3

(h) If I and J are two ideals of a ring R, define $f : R/I \cap J \to R/I \times R/J$ by $f(a + I \cap J) = (a + I, a + J)$ $\forall\ a \in R$. Show that f is well-defined isomorphism. 2+3

Unit - II

Answer ***any four*** questions.

3. (a) Prove that any linearly independent set of n vectors of a vector space V over R is a basis of V if $\dim V = n$. Show that R_3 can not be spanned with two vectors. 4+1

(b) Let V be a finite dimensional vector space over a field F and W be a subspace of V. Show that $\dim (V/W) = \dim V - \dim W$. 5

(c) If $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear mapping such that (1, 2, 3), (3, 0, 1) and (0, 3, 1) goes to (–3, 0, –2), (–5, 2, –2), (4, –1, 1) respectively, then show that T is an isomorphism. 5

(d) If $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be defined as $T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$, then find rank and nullity of T. 3+2

(e) Prove that the sets {(1, 1, 0, 0), (1, 0, 1, 1)} and {(2, –1, 3, 3), (0, 1, –1, –1)} generate same vector subspaces of $\mathbb{R}^4$. 5

(f) Let T be a linear map on $\mathbb{R}^3$ defined as $T(a, b, c) = (2a, 2a - 5b, 2b + c)$. Does T^{-1} exists? If exists, find T^{-1}. 2+3

(g) Prove that the linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, y + z, x + z)$ is one-one and onto. 5

(h) Show that in a vector space defined over $\mathbb{R}$, the vector addition is always commutative and it follows from other defined properties. 5